

GENERALIZED VARIETIES OF SUMS OF POWERS

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ABSTRACT. Let $X \subset \mathbb{P}^N$ be an irreducible, non-degenerate variety. The generalized variety of sums of powers $VSP_H^X(h)$ of X is the closure in the Hilbert scheme $\text{Hilb}_h(X)$ of the locus parametrizing collections of points $\{x_1, \dots, x_h\}$ such that the $(h-1)$ -plane $\langle x_1, \dots, x_h \rangle$ passes through a fixed general point $p \in \mathbb{P}^N$. When $X = V_d^n$ is a Veronese variety we recover the classical variety of sums of powers $VSP(F, h)$ parametrizing additive decompositions of a homogeneous polynomial as powers of linear forms. In this paper we study the birational behavior of $VSP_H^X(h)$. In particular we will show how some birational properties, such as rationality, unirationality and rational connectedness, of $VSP_H^X(h)$ are inherited from the birational geometry of variety X itself.

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INTRODUCTION

Let $X \subset \mathbb{P}^N$ be an irreducible, non-degenerate variety. The Zariski closure of the union of the linear spaces spanned by collections of h points on X is the h -secant variety $\text{Sec}_h(X)$ of X . Secant varieties are central objects in both classical algebraic geometry [CC], [Za], and applied mathematics [LM], [LO], [MR].

The abstract h -secant variety $\text{Sec}_h(X)$ is the Zariski closure of the set of couples (p, H) where $p \in \mathbb{P}^N$ and H is an $(h-1)$ -plane h -secant to X . In this paper we study the general fiber of the natural map $\pi_h : \text{Sec}_h(X) \rightarrow \mathbb{P}^N$. If $p \in \mathbb{P}^N$ is a general point such fiber parametrizes $(h-1)$ -planes h -secant to X passing through p .

Let V_d^n be the Veronese variety obtained as the image of the embedding $\nu_d^n : \mathbb{P}^n \rightarrow \mathbb{P}^N$ induced by $\mathcal{O}_{\mathbb{P}^n}(d)$. When $X = V_d^n$ the general fiber of $\pi_h : \text{Sec}_h(V_d^n) \rightarrow \mathbb{P}^N$ parametrizes decompositions of a general homogeneous polynomial $F \in k[x_0, \dots, x_n]$ as sums of powers of linear forms. In this case the fiber $\pi_h^{-1}(F)$ is called the *variety of sums of powers* of F and

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denoted by $VSP(F, h)$. The varieties $VSP(F, h)$ have been widely studied from both the biregular [IR], [Mu1], [Mu2], [RS] and the birational viewpoint [MMe]. The interest in these varieties increased after *S. Mukai* described the Fano 3-fold V_{22} and a polarized $K3$ surface of genus 20 as the varieties of sums of powers of general polynomials $F \in k[x_0, x_1, x_2]_4$ and $F \in k[x_0, x_1, x_2]_6$ respectively [Mu1], [Mu2]. Then many authors generalized Mukai's techniques to other polynomials [IR], [DK], [RS], [TZ]. See [Do] for a survey. Recently the variety of sums of powers of a polynomial $F \in k[x_0, \dots, x_5]_4$ has been used to construct particular divisors in the moduli space of cubic 4-folds [RV].

In this paper, in analogy with the classical case, we denote by $VSP_G^X(h)$ the general fiber of the map $\pi_h : \text{Sec}_h(X) \rightarrow \mathbb{P}^N$ and we call the varieties $VSP_G^X(h)$ *generalized varieties of sums of powers*, see Definition 1.5. The letter G reminds us that we are looking at $VSP_G^X(h)$ in the Grassmannian $\mathbb{G}(h-1, N)$. In Definition 1.1 we introduce the varieties $VSP_H^X(h)$ as a subvariety of the Hilbert scheme of points $\text{Hilb}_h(X)$. However, under a suitable numerical hypothesis $VSP_G^X(h)$ and $VSP_H^X(h)$ turn out to be birational, see Remark 1.6.

Our aim is to investigate the birational behavior of $VSP_H^X(h)$. More precisely we will show how some birational properties of $VSP_H^X(h)$ are inherited from the birational geometry of X itself.

In Section 2 we consider the case when X is a variety of minimal degree. That is an irreducible, non-degenerate variety $X \subset \mathbb{P}^N$ such that $\deg(X) = \text{codim}(X) + 1$. In this context our main result is Theorem 2.3.

Theorem. *Let $X \subset \mathbb{P}^N$ be a variety of minimal degree $\deg(X) = d$. Then $VSP_H^X(h)$ is irreducible for any $h \geq d$. Furthermore $VSP_H^X(h)$ is rational if $h = d$, and unirational for any $h \geq d$.*

In Section 3 we consider the case when the rational map

$$\begin{aligned} \chi : \quad VSP_H^X(h) &\dashrightarrow \mathbb{G}(h-2, N-1) \\ \{x_1, \dots, x_h\} &\longmapsto \langle x_1, \dots, x_h \rangle \end{aligned}$$

is dominant. Here the Grassmannian $\mathbb{G}(h-2, N-1)$ parametrizes $(h-1)$ -planes passing through a general point $p \in \mathbb{P}^N$. Then, by studying the general fiber of χ , we get Theorem 3.2.

Theorem. *Let $X \subset \mathbb{P}^N$ be an irreducible variety. Assume $h > N - \dim(X) + 1$ and that the general $(h-1)$ -dimensional linear section of X is rationally connected. Then the irreducible components of $VSP_H^X(h)$ are rationally connected.*

In Proposition 3.6, with a similar argument, we prove that the irreducible components of the classical varieties of sums of power $VSP(F, h)$ of a general homogeneous polynomial $F \in k[x_0, \dots, x_n]_d$ are rationally connected as soon as $h \geq \frac{d(N+1)-n}{d}$. Furthermore in Theorem 3.15 we obtain a generalization of [MMe, Theorem 4.1] by considering an arbitrary unirational variety instead of the Veronese variety.

Theorem. *Let $X \subset \mathbb{P}^N$ be a unirational variety. Assume that for some positive integer $k < n$ the number $\bar{h} = \frac{N}{k+1}$ is an integer and*

$$\frac{N+n+2}{n+1} \leq \bar{h} < N-n+1.$$

Then the irreducible components of $VSP_H^X(h)$ are rationally connected for $h \geq \bar{h}$.

Finally in Section 4 we consider the cases when there exists a canonical decomposition. This means that there exists a positive integer \bar{h} such that $\text{Sec}_{\bar{h}}(X) = \mathbb{P}^N$ and $VSP_H^X(\bar{h})$ is a single point, that is through a general point $p \in \text{Sec}_{\bar{h}}(X)$ passes exactly one $(\bar{h} - 1)$ -plane \bar{h} -secant to X . Our main result is Theorem 4.1.

Theorem. *Let $X \subset \mathbb{P}^N$ be an irreducible rational variety. Assume that there exists a positive integer \bar{h} such that $\text{Sec}_{\bar{h}}(X) = \mathbb{P}^N$ and $VSP_H^X(\bar{h})$ is a single point. Then $VSP_H^X(h)$ is unirational for any $h \geq \bar{h}$.*

Furthermore we study the uniqueness of the decomposition when $X = V_d^n$ is a Veronese variety. The first results in this direction are due to *J. J. Sylvester* [Sy], *D. Hilbert* [Hi], *H. W. Richmond* [Ri], and *F. Palatini* [Pa]. In the last few years this problem has been studied in [Me1] and [Me2]. As widely expected, the canonical decomposition very seldom exists [Me2, Theorem 1]. However it is known that a general homogeneous polynomial $F \in k[x_0, \dots, x_n]_d$ admits a canonical decomposition as a sum of d -th powers of linear forms in the following cases.

- $n = 1, d = 2h - 1$ [Sy],
- $n = d = 3, h = 5$ [Sy],
- $n = 2, d = 5, h = 7$ [Hi].

In Proposition 4.4 and Theorems 4.7, 4.9 we give very simple and geometrical proofs of these three facts. Furthermore in each one of the listed cases we give an algorithm to reconstruct the decomposition of a given polynomial.

1. NOTATION AND PRELIMINARIES

We work over an algebraically closed field k of characteristic zero.

Varieties of sums of powers and secant varieties. Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non-degenerate variety and let $\text{Hilb}_h(X)$ be the Hilbert scheme parametrizing zero-dimensional subschemes of X of length h .

Definition 1.1. Let $p \in \mathbb{P}^N$ be a general point. We define

$$VSP_H^X(p, h)^o := \{\{x_1, \dots, x_h\} \in \text{Hilb}_h(X) \mid p \in \langle x_1, \dots, x_h \rangle\} \subseteq \text{Hilb}_h(X),$$

and

$$VSP_H^X(p, h) := \overline{VSP_H^X(p, h)^o}.$$

by taking the closure of $VSP_H^X(p, h)^o$ in $\text{Hilb}_h(X)$. When there is no danger of confusion, we write simply $VSP_H^X(h)$ for $VSP_H^X(p, h)$.

Let $\nu_d^n : \mathbb{P}^n \rightarrow \mathbb{P}^{N(n,d)}$, with $N(n, d) = \binom{n+d}{d} - 1$ be the Veronese embedding induced by $\mathcal{O}_{\mathbb{P}^n}(d)$, and let $V_d^n = \nu_d^n(\mathbb{P}^n)$ be the corresponding Veronese variety. Note that when $X = V_d^n$ we recover the classical variety of sums of powers $VSP_H^X(h) = VSP(F, h)$ parametrizing additive decompositions of a general homogeneous polynomial $F \in k[x_0, \dots, x_n]_d$ as sum of d -powers of linear forms, see [Do].

Proposition 1.2. *Assume the general point $p \in \mathbb{P}^N$ to be contained in a $(h - 1)$ -linear space h -secant to X . Then the variety $VSP_H^X(h)$ has dimension*

$$\dim(VSP_H^X(h)) = h(n + 1) - N - 1.$$

Furthermore if $n = 2$ and X is a smooth surface then for p varying in an open Zariski subset of \mathbb{P}^N the varieties $VSP_H^X(h)$ are smooth and irreducible.

Proof. Consider the incidence variety

$$\begin{array}{ccc} \mathcal{I} = \{(Z, p) \mid Z \in VSP_H^X(h)\} & \subseteq & \text{Hilb}_h(X) \times \mathbb{P}^N \\ & \searrow \phi & \swarrow \psi \\ & \text{Hilb}_h(X) & \mathbb{P}^N \end{array}$$

The morphism ϕ is surjective and there exists an open subset $U \subseteq \text{Hilb}_h(X)$ such that for any $Z \in U$ the fiber $\phi^{-1}(Z)$ is isomorphic to the Grassmannian \mathbb{P}^{h-1} , so $\dim(\phi^{-1}(Z)) = h - 1$. The fibers of ψ are the varieties $VSP_H^X(h)$. Under our hypothesis the morphism ψ is dominant and

$$\dim(VSP_H^X(h)) = \dim(\mathcal{I}) - N = h(n + 1) - N - 1.$$

If $n = 2$ and X is a smooth surface then $\text{Hilb}_h(X)$ is smooth. The fibers of ϕ over U are open Zariski subsets. So \mathcal{I} is smooth and irreducible. Since the varieties $VSP_H^X(h)$ are the fibers of ψ we conclude that for p varying in an open Zariski subset of \mathbb{P}^N the varieties $VSP_H^X(h)$ are smooth and irreducible. \square

Now, we want to define another compactification of $VSP_H^X(h)^\circ$ in the Grassmannian $\mathbb{G}(h - 1, N)$. In order to do this we need to introduce secant varieties. Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non-degenerate variety and let

$$\Gamma_h(X) \subset X \times \cdots \times X \times \mathbb{G}(h - 1, N)$$

be the reduced closure of the graph of

$$\alpha : X \times \cdots \times X \dashrightarrow \mathbb{G}(h - 1, N),$$

taking h general points to their linear span $\langle x_1, \dots, x_h \rangle$. Now, $\Gamma_h(X)$ is irreducible and reduced of dimension hn . Let $\pi_2 : \Gamma_h(X) \rightarrow \mathbb{G}(h - 1, N)$ be the natural projection. We denote

$$\mathcal{S}_h(X) := \pi_2(\Gamma_h(X)) \subset \mathbb{G}(h - 1, N).$$

Note that $\mathcal{S}_h(X)$ is irreducible and reduced of dimension hn . Finally, let

$$\mathcal{I}_h = \{(x, \Lambda) \mid x \in \Lambda\} \subset \mathbb{P}^N \times \mathbb{G}(h - 1, N)$$

with the projections π_h and ψ_h onto the factors.

The *abstract h -secant variety* is the irreducible and reduced variety

$$\text{Sec}_h(X) := (\psi_h)^{-1}(\mathcal{S}_h(X)) \subset \mathcal{I}_h.$$

The *h -secant variety* is

$$\text{Sec}_h(X) := \pi_h(\text{Sec}_h(X)) \subset \mathbb{P}^N.$$

The variety $\text{Sec}_h(X)$ has dimension $(hn + h - 1)$ and a natural \mathbb{P}^{h-1} -bundle structure on $\mathcal{S}_h(X)$. The variety X is *h -defective* if $\dim \text{Sec}_h(X) < \min\{\dim \text{Sec}_h(X), N\}$.

Remark 1.3. Note that in Proposition 1.2 the assumption that the general point $p \in \mathbb{P}^N$ is contained in a $(h - 1)$ -linear space h -secant to X can be rephrased as $\text{Sec}_h(X) = \mathbb{P}^N$.

We need to extend these notions to the relative case. Let S be a noetherian scheme and let $X \rightarrow S$ be a scheme over S such that there exists a coherent sheaf E on S with a closed embedding of X in $\mathbb{P}(E) := \text{Proj } \text{Sym}_{\mathcal{O}_S}(E)$ over S . Equivalently, we can assume that there exists a relatively ample line bundle L on X over S .

There exists a scheme $\text{Grass}(h, E)$ that finely parametrizes locally free sub-sheaves of rank h of E . Furthermore, $\text{Grass}(h, E)$ is projective over S .

Now suppose that E is a rank $N+1$ vector bundle and the fiber of the morphism $\text{Grass}(h, E) \rightarrow S$ over a closed point $s \in S$ is the Grassmannian $\text{Grass}(h, E_s) \cong \mathbb{G}(h, N)$, where E_s is the fiber of E over $s \in S$. There is a well-defined rational map over S

$$\begin{array}{ccc} X \times_S \cdots \times_S X & \xrightarrow{\alpha} & \text{Grass}(h, E) \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

mapping (x_1, \dots, x_h) to the linear span $\langle x_1, \dots, x_h \rangle$. Note that since α is a map over S , we are taking $x_i \in X_s \subset \mathbb{P}(E_s) \cong \mathbb{P}^N$ for some $s \in S$. Take $\Gamma_h^S(X)$ to be the reduced closure of the graph of α in $X \times_S \cdots \times_S X \times_S \text{Grass}(h, E)$; then $\Gamma_h^S(X)$ is irreducible and reduced of dimension hn over S .

Let $\pi : \Gamma_h^S(X) \rightarrow \text{Grass}(h, E)$ be the projection, denoted by

$$\mathcal{S}_h^S(X) := \pi(\Gamma_h^S(X)) \subseteq \text{Grass}(h, E).$$

Again $\mathcal{S}_h^S(X)$ is irreducible and reduced of dimension hn over S , where $n = \dim_S(X)$. Now consider the incidence correspondence

$$\begin{array}{ccc} \mathcal{I}_h^S := \{(z, F) \mid z \in F\} \subseteq \mathbb{P}(E) \times_S \text{Grass}(h, E) & & \\ \pi_h \swarrow & & \searrow \psi_h \\ \mathbb{P}(E) & & \text{Grass}(h, E) \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Let $X \rightarrow S$ be an irreducible and reduced scheme over S , together with a closed embedding in $\mathbb{P}(E)$. The *abstract relative h -secant variety* of X over S is

$$\text{Sec}_h^S(X) := \psi_h^{-1}(\mathcal{S}_h^S(X)) \subseteq \mathcal{I}_h^S$$

and the *relative h -secant variety* of X over S is

$$\text{Sec}_h^S(X) := \pi_h(\text{Sec}_h^S(X)) \subseteq \mathbb{P}(E).$$

Remark 1.4. The scheme $\text{Sec}_h^S(X)$ naturally comes with a morphism $\text{Sec}_h^S(X) \rightarrow S$ whose fiber over a closed point $s \in S$ is the h -secant variety $\text{Sec}_h(X_s) \subseteq \mathbb{P}(E_s) \cong \mathbb{P}^N$ of the fiber X_s of $X \rightarrow S$ over $s \in S$.

Definition 1.5. Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety of dimension n and let $p \in \mathbb{P}^N$ be a general point. For $h+n < N+1$ consider the h -secant map $\pi_h : \text{Sec}_h(X) \rightarrow \mathbb{P}^N$ and define

$$VSP_G^X(p, h) := \pi_h^{-1}(p).$$

When there is no danger of confusion, we write simply $VSP_G^X(h)$ for $VSP_G^X(p, h)$.

Remark 1.6. Let us compare the two compactifications $VSP_H^X(h)$ and $VSP_G^X(h)$. A general point of $VSP_G^X(h)$ corresponds to a $(h-1)$ -linear space h -secant to X and passing through $p \in \mathbb{P}^N$. Clearly there is a dominant rational map

$$\begin{aligned} \tau : VSP_H^X(h) &\dashrightarrow VSP_G^X(h) \\ \{x_1, \dots, x_h\} &\longmapsto \langle x_1, \dots, x_h \rangle \end{aligned}$$

Furthermore if $h+n < N+1$ the general $(h-1)$ -linear space intersects X in a subscheme consisting of h distinct points, so $\tau : VSP_H^X(h) \dashrightarrow VSP_G^X(h)$ is birational.

Remark 1.7. The inequality $h+n < N+1$ of Definition 1.5 is not a restriction for our purposes. Indeed when we use the compactification $VSP_G^X(h)$ we begin by studying $VSP_G^X(\bar{h})$ for a particular value \bar{h} of h satisfying this inequality and then we extend our conclusions for $h \geq \bar{h}$ using Construction 3.8.

Rational connectedness. We say that a variety X is *rational chain connected* if there is a family of proper and connected algebraic curves $g : U \rightarrow Y$ whose geometric fibers have only rational components with a morphism $\nu : U \rightarrow X$ such that

$$\nu \times \nu : U \times_Y U \rightarrow X \times X$$

is dominant. Note that the image of $\nu \times \nu$ consists of pairs $(x_1, x_2) \in X$ such that $x_1, x_2 \in u(\pi^{-1}(y))$ for some $y \in Y$, where $\pi : U \times_Y U \rightarrow Y$ is the projection. We say that X is *rational connected* if there is a family of proper and connected algebraic curves $g : U \rightarrow Y$ whose geometric fibers are irreducible rational curves with morphism $\nu : U \rightarrow X$ such that $\nu \times \nu$ is dominant, see [Ko, Definition IV.3.2].

A proper variety X over an algebraically closed field is *rational chain connected* if there is a chain of rational curves through any two general points $x_1, x_2 \in X$. The variety X is *rational connected* if there is an irreducible rational curve through any two general points $x_1, x_2 \in X$. If X is smooth these two notions are indeed equivalent, see [Ko, Theorem IV.3.10]. This is clearly false when X is singular. For instance the cone C_E over an elliptic curve E is *rational chain connected* but it is not *rational connected*.

Furthermore rational connectedness is a birational property and indeed if X is *rational connected* and $X \dashrightarrow Y$ is a dominant rational map then Y is *rational connected*. On the other hand *rational chain connectedness* is not a birational property. For instance the cone C_E is *rational chain connected* by chains of length two but its resolution \tilde{C}_E is a fibration over E with fibers \mathbb{P}^1 so it is not *rational chain connected*. Finally for rational fibrations the rational connectedness of the base and of the general fiber translates in the rational connectedness of the variety itself.

Proposition 1.8. *Let $\phi : X \dashrightarrow Y$ be a dominant rational map. If Y and the general fiber of ϕ are *rational connected* then X is *rational connected*.*

Proof. Let us consider a resolution of the indeterminacy

$$\begin{array}{ccc} \tilde{X} & & \\ \pi \downarrow & \searrow \tilde{\phi} & \\ X & \xrightarrow{\phi} & Y \end{array}$$

of the rational map ϕ . Now, $\tilde{\phi} : \tilde{X} \rightarrow Y$ is a dominant morphism with rationally connected general fiber. Then, by [GHS, Corollary 1.3] \tilde{X} is rationally connected. Finally the morphism $\pi : \tilde{X} \rightarrow X$ is birational and X is rationally connected as well. \square

The rational connectedness of a projective variety $X \subset \mathbb{P}^N$ is related to the fact that X has low degree. For instance a smooth hypersurface $X \subset \mathbb{P}^N$ of degree d is rationally connected if and only if $d \leq N$. In this context we will need the following theorem.

Theorem 1.9. [MaM, Theorem 3.1] *Let $X \subset \mathbb{P}^N$ be a variety set theoretically defined by homogeneous polynomials G_i of degree d_i , for $i = 1, \dots, m$, and let $l \geq 2$ be an integer. If*

$$\sum_{i=1}^m d_i \leq \frac{N(l-1) + m}{l}$$

then X is rationally chain connected by chains of lines of length at most l . In particular if X is smooth and the above inequality is satisfied then X is rationally connected by rational curves of degree at most l .

Note that the variety X of Theorem 1.9 is a general projective variety, which does not need to be neither smooth nor a complete intersection.

2. VARIETIES OF MINIMAL DEGREE AND RATIONALITY

There is a lower bound on the degree of an irreducible, reduced and non-degenerate variety $X \subset \mathbb{P}^N$.

Proposition 2.1. *If $X \subset \mathbb{P}^N$ is an irreducible, reduced and non-degenerate variety, then $\deg(X) \geq \text{codim}(X) + 1$.*

Proof. If $\text{codim}(X) = 1$, being X non-degenerate, we have $\deg(X) \geq 2 = \text{codim}(X) + 1$. We proceed by induction on $\text{codim}(X)$. Let $x \in X$ be a general point, and

$$\pi_x : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$$

be the projection from x . The variety $Y = \overline{\pi_x(X)} \subset \mathbb{P}^{N-1}$ has degree $\deg(Y) = \deg(X) - 1$, and codimension $\text{codim}(Y) = \text{codim}(X) - 1$. By induction hypothesis we have $\deg(Y) \geq \text{codim}(Y) + 1$, which implies $\deg(X) \geq \text{codim}(X) + 1$. \square

Definition 2.2. We say that an irreducible, reduced and non-degenerate variety $X \subset \mathbb{P}^N$ is a *variety of minimal degree* if $\deg(X) = \text{codim}(X) + 1$.

If $\text{codim}(X) = 1$ then X is a quadric hypersurface, and then classified by its dimension and its singular locus. In higher codimension we have that if $X \subset \mathbb{P}^N$ is a singular variety of minimal degree, then X is a cone over a smooth such variety. If X is smooth and $\text{codim}(X) \geq 2$, then X is either a rational normal scroll or the Veronese surface $V_2^2 \subset \mathbb{P}^5$. For a survey on varieties of minimal degree see [EH].

Theorem 2.3. *Let $X \subset \mathbb{P}^N$ be a variety of minimal degree $\deg(X) = d$. Then $VSP_H^X(h)$ is irreducible for any $h \geq d$. Furthermore $VSP_H^X(h)$ is rational if $h = d$, and unirational for any $h \geq d$.*

Proof. Let $p \in \mathbb{P}^N$ be a general point. Since

$$\dim(X) + (d-1) = N - \operatorname{codim}(X) + d - 1 = N$$

a general $(d-1)$ -plane Λ through p intersects X in d distinct points $\Lambda \cap X = \{x_1, \dots, x_d\}$. Clearly $p \in \Lambda = \langle x_1, \dots, x_d \rangle$, and $\operatorname{Sec}_d(X) = \mathbb{P}^N$. The $(d-1)$ -planes in \mathbb{P}^N passing through p are parametrized by the Grassmannian $\mathbb{G}(N-d, N-1)$. Therefore we have a generically injective rational map

$$\begin{array}{ccc} \chi : & \mathbb{G}(N-d, N-1) & \dashrightarrow VSP_H^X(d) \\ & \Lambda & \longmapsto \Lambda \cap X \end{array}$$

Now, it is enough to observe that

$$\dim(\mathbb{G}(N-d, N-1)) = (N-d+1)(d-1) = n(d-1) = d(n+1) - N - 1 = \dim(VSP_H^X(d))$$

to conclude that $VSP_H^X(d)$ is irreducible and rational.

For $h > d$ consider the incidence variety

$$\begin{array}{ccc} Y = \{((x_1, \lambda_1), \dots, (x_{h-d}, \lambda_{h-d}), \Lambda) \mid p - \sum_{i=1}^{h-d} \lambda_i x_i \in \Lambda\} & \subseteq & (X \times \mathbb{P}^1)^{h-d} \times \mathbb{G}(d-1, N) \\ & \begin{array}{cc} \phi \swarrow & \searrow \psi \\ (X \times \mathbb{P}^1)^{h-d} & \mathbb{G}(d-1, N) \end{array} & \end{array}$$

The morphism $\phi : Y \rightarrow (X \times \mathbb{P}^1)^{h-d}$ is surjective and its fibers are isomorphic to the Grassmannian $\mathbb{G}(N-d, N-1)$. Then Y is irreducible. Note that $(X \times \mathbb{P}^1)^{h-d}$ is rational being X of minimal degree and hence rational. Then the variety Y is rational. Since χ is birational, for $((x_1, \lambda_1), \dots, (x_{h-d}, \lambda_{h-d}), \Lambda) \in Y$ general the intersection $\Lambda \cap X = \{\hat{x}_1, \dots, \hat{x}_d\}$ determines a decomposition $p - \sum_{i=1}^{h-d} \lambda_i x_i = \sum_{j=1}^d \hat{\lambda}_j \hat{x}_j$. The map

$$\begin{array}{ccc} \alpha : & Y & \dashrightarrow VSP_H^X(h) \\ & ((x_1, \lambda_1), \dots, (x_{h-d}, \lambda_{h-d}), \Lambda) & \longmapsto \{x_1, \dots, x_{h-d}, \hat{x}_1, \dots, \hat{x}_d\} \end{array}$$

is a generically finite, rational map, of degree $\binom{h}{h-d}$. Now, it is enough to observe that

$$\dim(Y) = (n+1)(h-d) + (N-d+1)(d-1) = h(n+1) - N - 1 = \dim(VSP_H^X(h))$$

to conclude that α is dominant. The variety $VSP_H^X(h)$ is finitely dominated by a rational variety, then it is irreducible and unirational. \square

Example 2.4. Let $Q \subset \mathbb{P}^3$ be a smooth quadric. Since any line through a general point $p \in \mathbb{P}^3$ cuts on Q a length two zero-dimensional subscheme the morphism

$$\chi : \mathbb{P}^2 \rightarrow VSP_H^Q(2)$$

is an injective regular morphism. By Proposition 1.2 $VSP_H^Q(2)$ is a smooth surface, so χ is an isomorphism and $VSP_H^Q(2) \cong \mathbb{P}^2$. More generally for a smooth quadric hypersurface $Q \subset \mathbb{P}^N$ we have a birational morphism

$$\mathbb{P}^{N-1} \rightarrow VSP_H^Q(2)$$

and $VSP_H^Q(2)$ is rational.

3. LOW DEGREE AND RATIONAL CONNECTEDNESS

In this section we will prove two results relating the rational connectedness of $VSP_H^X(h)$ to the fact that the base variety $X \subset \mathbb{P}^N$ has low degree. In the following we will always assume that $\text{Sec}_h(X) = \mathbb{P}^N$.

Convention 3.1. When we refer to a general decomposition we always consider the irreducible component of $VSP_H^X(h)^o$ containing it. We still denote by $VSP_H^X(h)$ its compactification.

Fibrations approach. Let us begin with the simple case of a non-degenerate hypersurface $X \subset \mathbb{P}^N$ of degree d . Let $\mathbb{G}(h-2, N-1)$ be the Grassmannian of $(h-1)$ -planes in \mathbb{P}^N passing through a general point $p \in \mathbb{P}^N$. We have a rational dominant map

$$\begin{aligned} \chi : \quad VSP_H^X(h) &\dashrightarrow \mathbb{G}(h-2, N-1) \\ \{x_1, \dots, x_h\} &\longmapsto \langle x_1, \dots, x_h \rangle \end{aligned}$$

Let $H \in \mathbb{G}(h-2, N-1)$ be a general point, $X_H = X \cap H$ and

$$X_H^h := \underbrace{X_h \times \dots \times X_h}_{h \text{ times}}.$$

The fiber $\chi^{-1}(H)$ is birational to X_H^h/S_h . Now, $X_H \subset H \cong \mathbb{P}^{h-1}$ is a hypersurface of degree d . Therefore X_H is rationally connected if and only if $d \leq h-1$. In this case X_H^h and $\chi^{-1}(H)$ are rationally connected as well. So χ is a rational dominant map on a rational variety with rationally connected general fiber. By Proposition 1.8 we conclude that if $d \leq h-1$ the $VSP_H^X(h)$ is rationally connected. In the following we generalize this argument.

Theorem 3.2. *Let $X \subset \mathbb{P}^N$ be an irreducible variety. Assume $h > N - \dim(X) + 1$ and that the general $(h-1)$ -dimensional linear section of X is rationally connected. Then the irreducible components of $VSP_H^X(h)$ are rationally connected.*

Proof. As before we have a rational map

$$\begin{aligned} \chi : \quad VSP_H^X(h) &\dashrightarrow \mathbb{G}(h-2, N-1) \\ \{x_1, \dots, x_h\} &\longmapsto \langle x_1, \dots, x_h \rangle \end{aligned}$$

Since $h > N - \dim(X) + 1$, if $H \in \mathbb{G}(h-2, N-1)$ is an $(h-1)$ -plane through p and $X_H = X \cap H$, then $\dim(X_H) \geq 1$ and χ is dominant.

Now, if $H \in \mathbb{G}(h-2, N-1)$ is general then $X_H \subset H \cong \mathbb{P}^{h-1}$ is a smooth, rationally connected variety. The general fiber $\chi^{-1}(H)$ is birational to X_H^h/S_h , so it is rationally connected as well. Finally, by Proposition 1.8 we conclude that $VSP_H^X(h)$ is rationally connected. \square

An immediate consequence of Theorem 3.2 is the following.

Proposition 3.3. *Let $X \subset \mathbb{P}^N$ be a smooth variety set theoretically defined by homogeneous polynomials G_i of degree d_i , for $i = 1, \dots, m$, and let $l \geq 2$ be an integer. Assume $h > N - \dim(X) + 1$. If*

$$\sum_{i=1}^m d_i \leq \frac{(h-1)(l-1) + m}{l}$$

then the irreducible components of $VSP_H^X(h)$ are rationally connected.

Furthermore if $X \subset \mathbb{P}^N$ is a smooth complete intersection of G_1, \dots, G_c hypersurfaces of degree d_i for $i = 1, \dots, c$ and

$$\sum_i^c d_i \leq h - 1$$

then the irreducible components of $VSP_H^X(h)$ are rationally connected.

Proof. If $H \in \mathbb{G}(h - 2, N - 1)$ is general then $X_H \subset H \cong \mathbb{P}^{h-1}$ is a smooth variety defined by homogeneous polynomials $\tilde{G}_i := G_{i|H}$ of degree d_i , for $i = 1, \dots, m$. By Theorem 1.9 under our numerical hypothesis the variety X_H is rationally connected. To conclude it is enough to apply Theorem 3.2.

If X is a smooth complete intersection then $X_H \subset \mathbb{P}^{h-1}$ is the smooth complete intersection of the hypersurfaces $\tilde{G}_i := G_{i|H}$ of degree d_i , for $i = 1, \dots, c$. The second statement follows from the fact that X_H is rationally connected if and only if $\sum_i^c d_i \leq h - 1$ and from Theorem 3.2. \square

Example 3.4. Let $X \subset \mathbb{P}^9$ be a complete intersection of two general hypersurfaces of degree two. Then the irreducible components of $VSP_H^X(h)$ are rationally connected for any $h \geq 5$.

Remark 3.5. Recall that we have a dominant rational map

$$\begin{aligned} \tau : \quad VSP_H^X(h) &\dashrightarrow VSP_G^X(h) \\ \{x_1, \dots, x_h\} &\longmapsto \langle x_1, \dots, x_h \rangle \end{aligned}$$

Therefore Theorem 3.2 holds for the compactification $VSP_G^X(h)$ as well.

By the argument used in the proof of Theorem 3.2 we get the following result for the classical varieties of sums of powers $VSP(F, h)$ when $X = V_d^n \subset \mathbb{P}^{N(n, d)}$ is a Veronese variety.

Proposition 3.6. *Let $VSP(F, h)$ be the variety of sums of powers of a general homogeneous polynomial $F \in k[x_0, \dots, x_n]_d$. If*

$$h \geq \frac{d(N + 1) - n}{d}$$

then the irreducible components of $VSP(F, h)$ are rationally connected.

Proof. The numerical hypothesis implies $h > N - n + 1$. Then the rational map

$$\begin{aligned} \chi : \quad VSP(F, h) &\dashrightarrow \mathbb{G}(h - 2, N - 1) \\ \{L_1, \dots, L_h\} &\longmapsto \langle L_1, \dots, L_h \rangle \end{aligned}$$

is dominant. In this case $X = V_d^n$ and the general fiber of $\chi^{-1}(H)$ is birational to X_H^h/S_h where X_H is a smooth complete intersection of $N - h + 1$ hypersurfaces of degree d in \mathbb{P}^n . Therefore

$$\omega_{X_H} \cong \mathcal{O}_{\mathbb{P}^n}((N - h + 1)d - n - 1).$$

To conclude it is enough to observe that under our numerical hypothesis X_H is Fano and therefore rationally connected and to apply Proposition 1.8. \square

Example 3.7. If $d = n = 3$, then $N = 19$. The bound of Proposition 3.6 is $h \geq 19$. For instance when $h = 19$ we have the fibration

$$\begin{aligned} \chi : VSP(F, 19) &\dashrightarrow \mathbb{G}(17, 18) \cong \mathbb{P}^{18} \\ \{L_1, \dots, L_{19}\} &\longmapsto \langle L_1, \dots, L_{19} \rangle \end{aligned}$$

whose general fiber is a cubic surface. Note that this case is not covered by [MMe, Theorem 4.1].

In the following table we work out some numbers which make Proposition 3.6 work. We denote by \bar{h} the smallest h for which Proposition 3.6 holds. Clearly this result makes sense for $n \gg d$.

| d | n | N | \bar{h} |
|-----|-----|----------|-----------|
| 3 | 100 | 176850 | 176818 |
| 3 | 150 | 585275 | 585226 |
| 4 | 200 | 70058750 | 70058701 |

Chains approach. In the following our aim is to prove the rational connectedness of another class of generalized varieties of sums of powers by extending the approach of [MMe, Section 4] from the Veronese varieties to arbitrary unirational varieties. In this section we will use the notion of very general point. Therefore we assume that ground field k is uncountable.

Construction 3.8. Assume $VSP_H^X(h)$ to be non-empty, and let $\{x_1, \dots, x_h\} \in VSP_H^X(h)$ be a general point. Then there exists a general point $p \in \mathbb{P}^N$ such that

$$p = \sum_{i=1}^h \lambda_i x_i.$$

The point $p - \lambda_1 x_1$ is general as well, and we get a generically injective rational map

$$\begin{aligned} VSP_H^X(h-1) &\dashrightarrow VSP_H^X(h) \\ \{y_1, \dots, y_{h-1}\} &\longmapsto \{x_1, y_1, \dots, y_{h-1}\} \end{aligned}$$

This construction yields a stratification

$$VSP_H^X(h-r) \subset VSP_H^X(h-r+1) \subset \dots \subset VSP_H^X(h-1) \subset VSP_H^X(h).$$

Proposition 3.9. Let $X \subset \mathbb{P}^N$ be a non-degenerate variety of dimension n . If

$$h \geq \frac{(N+1)}{n+1} + 2$$

then two very general points of $VSP_H^X(h)$ are joined by a chain, of at most length three, of $VSP_H^X(h-1)$. If V_i are the elements of this chain and $q \in V_i \cap V_j$ is a general point, then we can assume q to be a smooth point in V_i, V_j and $VSP_H^X(h)$.

Proof. Let $x = \{x_i\}, y = \{y_i\} \in VSP_H^X(h)$ be two very general points, and write

$$p = \sum_{i=1}^h \lambda_i x_i = \sum_{i=1}^h \gamma_i y_i.$$

Let $z \in VSP_H^X(p - \lambda_1 x_1, h-1)$ be a general point associated to the decomposition

$$p - \lambda_1 x_1 = \sum_{i=2}^h \alpha_i z_i.$$

Let $\nu : Z \rightarrow VSP_H^X(h)$ be a resolution of singularities. Since x and y are two very general points we can assume that

- $\nu^{-1}(VSP_H^X(p - \lambda_1 x_1, h - 1))$ and $\nu^{-1}(VSP_H^X(p_j - \gamma_1 y_1, h - 1))$ belong to the same irreducible component of $\text{Hilb}(Z)$.
- ν is an isomorphism in a neighborhood of q .

Since $z \in VSP_H^X(h)$ is associated to $p = \lambda_1 x_1 + \sum_{i=2}^h \alpha_i z_i$ we have

$$z \in VSP_H^X(p - \lambda_1 x_1, h - 1) \cap VSP_H^X(p - \alpha_2 z_2, h - 1).$$

Under our numerical hypothesis we have

$$\dim(VSP_H^X(p - \alpha_2 z_2, h - 1)) \geq \text{codim}_{VSP_H^X(h)}(VSP_H^X(p - \alpha_2 z_2, h - 1)),$$

and we conclude that

$$VSP_H^X(p - \alpha_2 z_2, h - 1) \cap VSP_H^X(p - \gamma_1 y_1, h - 1) \neq \emptyset.$$

Furthermore the general point of this intersection is a smooth point of $VSP_H^X(p - \alpha_2 z_2, h - 1)$, $VSP_H^X(p - \gamma_1 y_1, h - 1)$ and $VSP_H^X(h)$. Finally we can join two general points of $VSP_H^X(h)$ by a chain of length at most three of $VSP_H^X(h - 1)$. \square

In what follows our aim is to generalize [MMe, Theorem 4.1] taking an arbitrary unirational variety $X \subset \mathbb{P}^N$ instead of the Veronese variety $V_d^n \subset \mathbb{P}^{N(n,d)}$. The first step of our argument is to prove that a unirational variety can be covered by rational subvarieties in a suitable sense.

Proposition 3.10. *Let X be a unirational variety. For any triple of integers (a, b, c) , with $0 < c < n$, there is a rationally connected variety $V_{a,b,c}^n \subset \text{Hilb}(X)$ with the following properties:*

- a general point in $V_{a,b,c}^n$ represents a rational subvariety of X of codimension c ;
- for a general $Z \subset X$ reduced zero dimensional scheme of length $l \leq b$, there is a rationally connected subvariety $V_{Z,c} \subset V_{a,b,c}^n$ of dimension at least a , whose general element $[Y] \in V_{Z,c}$ represents a rational subvariety of X of codimension c containing Z .

Proof. Since X is unirational there is a generically finite, dominant map $\phi : \mathbb{P}^n \dashrightarrow X$. For any Hilbert polynomial $P \in \mathbb{Q}[z]$ the map ϕ induces a generically finite rational map

$$\begin{array}{ccc} \chi : \text{Hilb}_P(\mathbb{P}^n) & \dashrightarrow & \text{Hilb}_Q(X) \\ Z & \longmapsto & \phi(Z) \end{array}$$

We prove the statement by induction on c . Assume $c = 1$, and consider an equation of the form

$$Y = (x_n A(x_0, \dots, x_{n-1})_{d-1} + B(x_0, \dots, x_{n-1})_d = 0),$$

then, for A and B general, $Y \subset \mathbb{P}^n$ is a rational hypersurface of degree d with a unique singular point of multiplicity $d - 1$ at the point $[0, \dots, 0, 1]$. Take A and B general. Let $\overline{Y} := \overline{\phi(Y)}$ be the closure of the image of Y in X . If $\overline{y} \in \overline{Y}$ is a general point the fiber $\phi^{-1}(\overline{y})$ intersects Y in a point, that is $\phi|_Y : Y \rightarrow \overline{Y}$ is birational.

Fix $d > ab$ and let $W_{a,b,1}^n \subset \mathbb{P}(k[x_0, \dots, x_n]_d)$ be the linear span of these hypersurfaces. We take $V_{a,b,1}^n := \chi(W_{a,b,1}^n)$. Let $Z = \{x_1, \dots, x_l\} \subset X$ be a zero dimensional subscheme of length $l \leq b$, and take $p_i \in \phi^{-1}(x_i)$ for $i = 1, \dots, l$.

For any triple $(a, b, 1)$ consider $W_{Z,1} \subset W_{a,b,1}^n$ as the sublinear system of hypersurfaces containing $\{p_1, \dots, p_l\}$. Now take $V_{Z,1} := \chi(W_{Z,1})$. Then on a general point $[Y] \in W_{Z,1}$ the map ϕ restricts to a birational map and a general point of $V_{Z,1}$ parametrizes a rational subvariety of codimension 1 in X containing Z .

Assume, by induction, that $W_{a,b,i-1}^n \subset \text{Hilb}(\mathbb{P}^{n-1})$ exist for any n and b . Define, for $i \geq 2$,

$$\widetilde{W}_{a,b,i}^n := W_{a,b,1}^n \times W_{a,b,i-1}^{n-1} \subset \text{Hilb}(\mathbb{P}^n) \times \text{Hilb}(\mathbb{P}^{n-1}).$$

Let $[Y]$ be a general point in $W_{a,b,1}^n$. By construction Y has a point of multiplicity $d-1$ at the point $[0, \dots, 0, 1] \in \mathbb{P}^n$. Then the projection $\pi_{[0, \dots, 0, 1]} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ restricts to a birational map $\varphi_Y : Y \dashrightarrow \mathbb{P}^{n-1}$. Hence we may associate the general element $([Y], [S]) \in \{[Y]\} \times W_{a,b,i-1}^{n-1}$ to the codimension i subvariety $\varphi_Y^{-1}(S) \subset \mathbb{P}^n$. By [Ko, Proposition I.6.6.1] this yields a rational map

$$\begin{aligned} \alpha : \quad \widetilde{W}_{a,b,i}^n &\dashrightarrow \text{Hilb}(\mathbb{P}^n) \\ (Y, S) &\longmapsto \varphi_Y^{-1}(S) \end{aligned}$$

Let $W_{a,b,i}^n := \overline{\alpha(\widetilde{W}_{a,b,i}^n)} \subset \text{Hilb}(\mathbb{P}^n)$. For any Z we may define

$$\widetilde{W}_{Z,i} := W_{Z,1} \times W_{\pi_{[1,0,\dots,0]}(Z),i-1},$$

and as above $W_{Z,i} = \overline{\alpha(\widetilde{W}_{Z,i})}$.

By construction a general point of $W_{a,b,c}^n$ is the inverse image of a rational subvariety of codimension $c-1$ in \mathbb{P}^{n-1} via the projection from the singular point of a general rational hypersurface in $W_{a,b,1}^n$. Then on the general subvariety parametrized by $W_{a,b,c}^n$ and $\widetilde{W}_{Z,c}$ the map ϕ restricts to a birational map. We take $V_{a,b,c}^n := \chi(W_{a,b,c}^n)$ and $V_{Z,c} := \chi(\widetilde{W}_{Z,c})$. The varieties $V_{a,b,c}^n$ and $V_{Z,c}$ are dominated by rationally connected varieties, so they are rationally connected as well. \square

Remark 3.11. Let $X \subset \mathbb{P}^N$ be a rational, non-degenerate variety of dimension n , and let $\phi : \mathbb{P}^n \dashrightarrow X$ be a birational map. The linear system $\mathcal{H} = \phi^* \mathcal{O}_{\mathbb{P}^N}(1)$ is a subsystem of $\mathcal{O}_{\mathbb{P}^n}(d)$ for some integer d . We can embed \mathbb{P}^n via the Veronese embedding ν_d^n in $\mathbb{P}^{N(n,d)}$. The variety X is a birational projection

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\nu_d^n} & V_d^n \subset \mathbb{P}^{N(n,d)} \\ & \searrow & \downarrow \pi \\ & & X \subset \mathbb{P}^N \end{array}$$

of V_d^n . This means that any rational variety can be seen as a birational projection of a suitable Veronese variety.

Let $X \subset \mathbb{P}^N$ be a unirational variety of dimension n . In the following, by Convention 3.1 we assume that $VSP_G^X(h)$ is irreducible. Let us assume there exists an integer k such that

- $0 < k < n$,
- $\bar{h} = \frac{N}{k+1}$ is an integer,
- $\bar{h} + n < N + 1$.

Remark 3.12. The last condition is required in Definition 1.5 because we are working with the compactification $VSP_G^X(h)$. Note that the inequality $\bar{h} + n < N + 1$ means that $X \subset \mathbb{P}^N$ has big codimension. On the other hand low codimension varieties are studied using the fibrations approach of Theorem 3.2. Finally, note that when $X = V_d^n$ is a Veronese variety the inequality $\bar{h} + n < N + 1$ is automatically satisfied.

Let $H_x = \langle x_1, \dots, x_{\bar{h}} \rangle$ and $H_y = \langle y_1, \dots, y_{\bar{h}} \rangle$ be two general points of $VSP_G^X(h)$. In the notation of Proposition 3.10 let us consider $V_{a, 2\bar{h}, n-k}^n \subset \text{Hilb}(X)$ for $a \gg 0$ and let $Y \in V_{a, 2\bar{h}, n-k}^n$ be a general point. Recall that:

- Y is a rational subvariety of X of codimension $n - k$,
- if $Z = \{x_1, \dots, x_{\bar{h}}, y_1, \dots, y_{\bar{h}}\}$ then there exists a rationally connected subvariety $V_{Z, n-k} \subset V_{a, 2\bar{h}, n-k}^n$ of dimension at least a whose general point represents a rational subvariety of X of codimension $n - k$ containing Z .

Lemma 3.13. *The \bar{h} -secant variety $\text{Sec}_{\bar{h}}(Y) \subset \mathbb{P}^N$ is a hypersurface. Furthermore through a general point of $\text{Sec}_{\bar{h}}(Y)$ there is a unique $(\bar{h} - 1)$ -plane \bar{h} -secant to Y and $\text{Sing}(\text{Sec}_{\bar{h}}(Y))$ has codimension one in $\text{Sec}_{\bar{h}}(Y)$.*

Proof. Since Y is rational, by Remark 3.11, we can see Y as a birational projection $\pi : V_d^k \dashrightarrow Y$ of a Veronese variety $V_d^k \subset \mathbb{P}^{N(k,d)}$ for $d \gg \bar{h}$. We denote by $\mathcal{B} \subset V_d^k$ the indeterminacy locus of π .

Let $z \in \text{Sec}_{\bar{h}}(V_d^k)$ be a general point. Assume $z \in \langle x_1, \dots, x_{\bar{h}} \rangle \cap \langle y_1, \dots, y_{\bar{h}} \rangle$. By Terracini's lemma [CC, Theorem 1.1] we have

$$\mathbb{T}_z \text{Sec}_{\bar{h}}(V_d^k) = \langle \mathbb{T}_{x_1} V_d^k, \dots, \mathbb{T}_{x_{\bar{h}}} V_d^k \rangle = \langle \mathbb{T}_{y_1} V_d^k, \dots, \mathbb{T}_{y_{\bar{h}}} V_d^k \rangle.$$

Then the general hyperplane section singular at $\{x_1, \dots, x_{\bar{h}}\}$ is singular at $\{y_1, \dots, y_{\bar{h}}\}$. Since $\text{codim}(\text{Sec}_{\bar{h}}(V_d^k)) \geq k + 1$ by [Me2, Corollary 4.5] V_d^k is not \bar{h} -weakly defective. Therefore by [CC, Theorem 1.4] the general hyperplane section singular at $\{x_1, \dots, x_{\bar{h}}\}$ is singular only at $\{x_1, \dots, x_{\bar{h}}\}$. So $\{x_1, \dots, x_{\bar{h}}\} = \{y_1, \dots, y_{\bar{h}}\}$ and through a general point of $\text{Sec}_{\bar{h}}(V_d^k)$ there is a unique $(\bar{h} - 1)$ -plane \bar{h} -secant to V_d^k . We conclude that

$$\dim(\text{Sec}_{\bar{h}}(V_d^k)) = k\bar{h} + \bar{h} - 1 = N - 1.$$

Let us assume that $\pi : \text{Sec}_{\bar{h}}(V_d^k) \dashrightarrow \text{Sec}_{\bar{h}}(Y)$ is not birational. The same argument of the first part of the proof shows that V_d^k is \bar{h} -weakly defective. This contradicts [Me2, Corollary 4.5]. Therefore $\text{Sec}_{\bar{h}}(Y)$ is a birational projection of $\text{Sec}_{\bar{h}}(V_d^k)$. So through a general point of $\text{Sec}_{\bar{h}}(Y)$ there is a unique $(\bar{h} - 1)$ -plane \bar{h} -secant to Y and $\dim(\text{Sec}_{\bar{h}}(Y)) = N - 1$.

Now, for a general $Y \in V_{a, 2\bar{h}, n-k}^n$ we have $\langle \text{Sec}_{\bar{h}}(Y) \cap \langle \mathcal{B} \rangle \rangle \subsetneq \langle \mathcal{B} \rangle$. Then we can factorize π as a composition of projections

$$\begin{array}{ccc} \mathbb{P}^{N(k,d)} & & \\ \pi_1 \downarrow & \searrow \pi & \\ \mathbb{P}^{N+1} & \xrightarrow{\pi_2} & \mathbb{P}^N \end{array}$$

where π_2 is a projection from a general point of \mathbb{P}^{N+1} . We know that $\text{Sec}_2(\pi_1(\text{Sec}_{\bar{h}}(V_d^k))) = \mathbb{P}^{N+1}$. Therefore the dimension of the singular locus of $\text{Sec}_{\bar{h}}(Y)$ is the dimension of the

space of secant lines to $\pi_1(\text{Sec}_{\bar{h}}(V_d^k))$ passing through p . That is

$$\dim(VSP_H^{\pi_1(\text{Sec}_{\bar{h}}(V_d^k))}(2)) = 2(N - 1 + 1) - (N + 1) - 1 = N - 2.$$

□

Let m be the degree of the hypersurface $\text{Sec}_{\bar{h}}(Y) \subset \mathbb{P}^N$.

Lemma 3.14. *There exists a generically injective rational map*

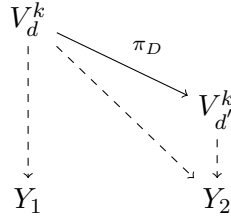
$$\begin{array}{ccc} \alpha : & V_{a,2\bar{h},n-k}^n & \dashrightarrow \mathbb{P}(k[z_0, \dots, z_N]_m) \\ & Y & \longmapsto \text{Sec}_{\bar{h}}(Y) \end{array}$$

Proof. Let $Y \in V_{a,2\bar{h},n-k}^n$ be a general point. By Lemma 3.13 we have

$$\text{Sec}_{\bar{h}}(Y) \in \mathbb{P}(k[z_0, \dots, z_N]_m).$$

Furthermore by Remark 1.4 assigning to a general point $Y \in V_{a,2\bar{h},n-k}^n$ its \bar{h} -secant variety $\text{Sec}_{\bar{h}}(Y)$ gives rise to a well defined rational map.

Now, we want to prove that α is generically injective. Let $Y_1 \in V_{a,2\bar{h},n-k}^n$ be a general point and let $Y_2 \in V_{a,2\bar{h},n-k}^n$ such that $\alpha(Y_1) = \text{Sec}_{\bar{h}}(Y_1) = \text{Sec}_{\bar{h}}(Y_2) = \alpha(Y_2)$. Let us denote $S := \text{Sec}_{\bar{h}}(Y_1) = \text{Sec}_{\bar{h}}(Y_2)$. Now, Y_1 and Y_2 are both rational and by Remark 3.11 we can see them as projections of two Veronese variety V_d^k and $V_{d'}^k$ for $d, d' \gg 0$.



We can assume $d \geq d'$. If $d > d'$ we can consider a general divisor $D \in |\mathcal{O}_{\mathbb{P}^k}(d - d')|$ and interpret $\pi_D : V_d^k \rightarrow V_{d'}^k$ as the projection from $\langle \nu_{k,d}(D) \rangle$. Note that π_D is well defined on V_d^k . This means that we can interpret both Y_1 and Y_2 as projections of the same Veronese variety V_d^k . We summarize the situation in the following diagram

$$\begin{array}{ccc} & V_d^k \subset \mathbb{P}^{N(n,d)} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ Y_1 \subset \mathbb{P}^N & & Y_2 \subset \mathbb{P}^N \end{array}$$

where π_1 and π_2 are projections from linear spaces Λ_1 and Λ_2 respectively. We get two projections

$$\pi_1 : \text{Sec}_{\bar{h}}(V_d^k) \dashrightarrow S, \quad \pi_2 : \text{Sec}_{\bar{h}}(V_d^k) \dashrightarrow S$$

and the composition $\gamma = \pi_2 \circ \pi_1^{-1}$ is a birational automorphism of S . By Lemma 3.13 $\text{Sing}(S)$ has codimension one. So γ is defined on the general point of $\text{Sing}(S)$. If $u \in \text{Sing}(S)$ is a general point and $v, w \in \pi_1^{-1}(u)$ are two general points then $\pi_2(v) = \pi_2(w) \in \text{Sing}(S)$. Therefore the line $\langle v, w \rangle$ intersects both Λ_1 and Λ_2 . Furthermore $\Lambda_1 \cap \langle v, w \rangle = \Lambda_2 \cap \langle v, w \rangle$. Proceeding recursively we conclude that $\Lambda_1 = \Lambda_2$ and therefore $Y_1 = Y_2$. □

Now we are ready to prove the following theorem.

Theorem 3.15. *Let $X \subset \mathbb{P}^N$ be a unirational variety. Assume that for some positive integer $k < n$ the number $\bar{h} = \frac{N}{k+1}$ is an integer and*

$$\frac{N+n+2}{n+1} \leq \bar{h} < N-n+1.$$

Then the irreducible components of $VSP_H^X(h)$ are rationally connected for $h \geq \bar{h}$.

Proof. We prove the statement for $h = \bar{h}$. Note that $\bar{h} \geq \frac{N+n+2}{n+1}$ implies

$$\bar{h} + 1 \geq \frac{N+1}{n+1} + 2.$$

Therefore in order to conclude for $h \geq \bar{h}$ we can apply Proposition 3.9.

Let $V = \overline{\alpha(V_{a,2\bar{h},n-k}^n)}$ be the closure of the image of $V_{a,2\bar{h},n-k}^n$ and let H_p be the hyperplane parametrizing hypersurfaces in $\mathbb{P}(k[z_0, \dots, z_N]_m)$ passing through the general point $p \in \mathbb{P}^N$. We consider the intersection $V_p = V \cap H_p$ parametrizing \bar{h} -secant varieties through p . By Lemma 3.13 through $p \in \text{Sec}_{\bar{h}}(Y)$ there is a unique $(\bar{h} - 1)$ -plane H_p^Y which is \bar{h} -secant to Y . Then we can define the rational map

$$\begin{aligned} \beta : \quad V_p & \dashrightarrow VSP_G^X(\bar{h}) \\ \text{Sec}_{\bar{h}}(Y) & \mapsto H_p^Y \end{aligned}$$

By Proposition 3.10 a general \bar{h} -secant linear space to X is \bar{h} -secant to some $Y \in V_{a,2\bar{h},n-k}^n$. Then the map β is dominant. Let $H_x = \langle x_1, \dots, x_{\bar{h}} \rangle$ and $H_y = \langle y_1, \dots, y_{\bar{h}} \rangle$ be two general points of $VSP_G^X(h)$. By Proposition 3.10 the varieties $V_{\{x_1, \dots, x_{\bar{h}}\}, n-k}$ and $V_{\{y_1, \dots, y_{\bar{h}}\}, n-k}$ are rationally connected. Furthermore we have

$$\alpha(V_{\{x_1, \dots, x_{\bar{h}}\}, n-k}) \subseteq \overline{\beta^{-1}(H_x)}$$

$$\alpha(V_{\{y_1, \dots, y_{\bar{h}}\}, n-k}) \subseteq \overline{\beta^{-1}(H_y)}$$

and

$$\alpha(V_{\{x_1, \dots, x_{\bar{h}}, y_1, \dots, y_{\bar{h}}\}, n-k}) \subseteq \overline{\beta^{-1}(H_x)} \cap \overline{\beta^{-1}(H_y)}.$$

Therefore V_p is rationally chain connected by chains of two rational curves intersecting in a general point of $\alpha(V_{\{y_1, \dots, y_{\bar{h}}\}, n-k})$.

Now, let $M_p \subset V_p$ be the variety parametrizing \bar{h} -secant varieties having more than one \bar{h} -secant $(\bar{h} - 1)$ -plane through p . By Lemma 3.13 a general $\text{Sec}_{\bar{h}}(Y) \in V$ is singular in codimension one. Therefore M_p has codimension two in V and since $M_p \subset V_p \subset V$ we have

$$\text{codim}_{V_p} M_p = 1.$$

Assume that the general point of $\text{Sing}(\text{Sec}_{\bar{h}}(Y))$ is of multiplicity $t \geq 2$. By Terracini's lemma [CC, Theorem 1.1] there are t \bar{h} -secant $(\bar{h} - 1)$ -planes through a general point of $\text{Sing}(\text{Sec}_{\bar{h}}(Y))$. In particular there exist two zero dimensional subschemes $\{z_1, \dots, z_{\bar{h}}\}$ and $\{w_1, \dots, w_{\bar{h}}\}$ such that

$$\text{Sec}_{\bar{h}}(Y) \in \alpha(V_{\{z_1, \dots, z_{\bar{h}}, w_1, \dots, w_{\bar{h}}\}, n-k}).$$

Therefore V_p is rationally chain connected by chains of rational curves intersecting in general points of M_p . Let us consider the normalization

$$\nu : \widetilde{V}_p \rightarrow V_p$$

and the intersection

$$\mathcal{I}_{x,y} = \alpha(V_{\{x_1, \dots, x_{\bar{h}}\}, n-k}) \cap \alpha(V_{\{y_1, \dots, y_{\bar{h}}\}, n-k}).$$

Then $\dim(\mathcal{I}_{x,y}) \geq a$. Let $\widetilde{M}_p = \nu^{-1}(M_p)$. We have that $\nu|_{\widetilde{M}_p}$ is a finite and étale morphism outside of a codimension one set K . For any point $q \in M_p \setminus K$ there is an étale open neighborhood U_q such that $\nu|_{\widetilde{M}_p}$ restricted to $\nu|_{\widetilde{M}_p}^{-1}(U_q)$ is finite and étale. Now, in the étale topology there exists an open subset $B \subset M_p$ such that $K \subset B$. The complement B^c is compact. Then we can cover B^c by finitely many $\{U_{q_i}\}_{i=1, \dots, r}$. Now, the map $\nu|_{\widetilde{M}_p}$ restricted to $\nu|_{\widetilde{M}_p}^{-1}(U_{q_i})$ is étale. Since $\{x_1, \dots, x_{\bar{h}}\}$ and $\{y_1, \dots, y_{\bar{h}}\}$ are general, V is irreducible and the $\{U_{q_i}\}_{i=1, \dots, r}$ are finitely many we have

$$\dim(\nu|_{\widetilde{M}_p}^{-1}(V_{\{x_1, \dots, x_{\bar{h}}\}, n-k}) \cap \nu|_{\widetilde{M}_p}^{-1}(V_{\{y_1, \dots, y_{\bar{h}}\}, n-k})) > 0.$$

Therefore \widetilde{V}_p is rationally chain connected by chains of rational curves through general points of $\nu^{-1}(M_p)$. Then \widetilde{V}_p and V_p are rationally connected as well. Finally, since the rational map β is dominant $VSP_H^X(\bar{h})$ is rationally connected. \square

Remark 3.16. When $X = V_d^n$ is the Veronese variety, from Theorem 3.15 we recover [MMe, Theorem 4.1]. In [IR] A. Iliev and K. Ranestad proved that if $X = V_3^5$ then $VSP_H^X(10)$ is a Hyperkähler manifold deformation equivalent to the Hilbert square of a K3 surface of genus 8. In particular $VSP_H^X(10)$ can not be rationally connected. In this case we have $N(n, d) = \binom{n+d}{n} - 1 = 55$, so $k + 1 = 5$, and Theorem 3.15 holds for $h \geq 11$.

In what follows we work out some numbers which make Theorem 3.15 work. We denote by \bar{h} the smallest h for which Theorem 3.15 holds.

Grassmannians. The Grassmannian $\mathbb{G}(r, n)$ parametrizing r -linear subspaces of \mathbb{P}^n is a rational homogeneous variety of dimension $(r+1)(n-r)$, and has a natural embedding

$$\mathbb{G}(r, n) \hookrightarrow \mathbb{P}^N,$$

with $N = \binom{n+1}{r+1} - 1$, called the Plücker embedding.

| r | n | $\dim(\mathbb{G}(r, n))$ | N | k | \bar{h} |
|-----|-----|--------------------------|-----|-----|-----------|
| 1 | 4 | 6 | 9 | 2 | 3 |
| 1 | 5 | 8 | 14 | 6 | 3 |
| 2 | 6 | 12 | 34 | 1 | 17 |
| 2 | 7 | 15 | 55 | 10 | 5 |
| 3 | 8 | 20 | 125 | 4 | 25 |

Segre-Veronese Varieties. Combining the Segre and the Veronese embeddings we can define the Segre-Veronese embedding

$$\psi : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^N,$$

with $N = \binom{a+n}{n} \binom{b+m}{m} - 1$, using the sheaf $\mathcal{O}_{\mathbb{P}^n}(a)$ on \mathbb{P}^n and the sheaf $\mathcal{O}_{\mathbb{P}^m}(b)$ on \mathbb{P}^m . Let $SV_{a,b}^{n,m} = \psi(\mathbb{P}^n \times \mathbb{P}^m)$ be the Segre-Veronese variety.

| n | m | a | b | $\dim(SV_{a,b}^{n,m})$ | N | k | \bar{h} |
|-----|-----|-----|-----|------------------------|------|-----|-----------|
| 2 | 3 | 1 | 3 | 5 | 39 | 2 | 13 |
| 4 | 4 | 2 | 3 | 8 | 524 | 3 | 131 |
| 4 | 4 | 3 | 3 | 8 | 1224 | 3 | 153 |
| 5 | 5 | 3 | 3 | 10 | 3135 | 4 | 627 |
| 5 | 5 | 3 | 4 | 10 | 7055 | 4 | 1411 |

4. CANONICAL DECOMPOSITIONS AND UNIRATIONALITY

In this section we consider the case when there exists a positive integer \bar{h} such that $\text{Sec}_{\bar{h}}(X) = \mathbb{P}^N$ and $VSP_H^X(\bar{h})$ is a single point, that is through a general point $p \in \text{Sec}_{\bar{h}}(X)$ passes exactly one $(\bar{h} - 1)$ -plane \bar{h} -secant to X . In the following we prove that the existence of such a canonical decomposition yields the unirationality of $VSP_H^X(h)$ for $h \geq \bar{h}$.

Theorem 4.1. *Let $X \subset \mathbb{P}^N$ be a rational variety. Assume that there exists a positive integer \bar{h} such that $\text{Sec}_{\bar{h}}(X) = \mathbb{P}^N$ and $VSP_H^X(\bar{h})$ is a single point. Then $VSP_H^X(h)$ is unirational for any $h \geq \bar{h}$.*

Proof. As usual, let us fix a general point $p \in \mathbb{P}^N$. For any $h > \bar{h}$ let us consider the incidence variety

$$\mathcal{I} = \{ \{x_1, \dots, x_{h-\bar{h}}, q\} \mid q \in \langle p, x_1, \dots, x_{h-\bar{h}} \rangle \} \subseteq (X)^{h-\bar{h}} \times \mathbb{P}^N$$

$\begin{array}{ccc} & \phi & \psi \\ & \swarrow & \searrow \\ (X)^{h-\bar{h}} & & \mathbb{P}^N \end{array}$

Now, X is rational, ϕ is dominant and its general fiber is a linear subspace of dimension $h - \bar{h}$ of \mathbb{P}^N . Therefore \mathcal{I} is rational as well. By hypothesis there exists a unique zero dimensional subscheme $\{y_1, \dots, y_{\bar{h}}\}$ spanning a $(\bar{h} - 1)$ -plane containing q . Furthermore $q \in \langle x_1, \dots, x_{h-\bar{h}}, p \rangle$ and $q \in \langle y_1, \dots, y_{\bar{h}} \rangle$ imply

$$p \in \langle y_1, \dots, y_{\bar{h}}, x_1, \dots, x_{h-\bar{h}} \rangle.$$

Therefore we have the generically finite rational map

$$\chi : \begin{array}{ccc} \mathcal{I} & \dashrightarrow & VSP_H^X(h) \\ \{x_1, \dots, x_{h-\bar{h}}\} & \mapsto & \{y_1, \dots, y_{\bar{h}}, x_1, \dots, x_{h-\bar{h}}\} \end{array}$$

Under our hypothesis on $\text{Sec}_{\bar{h}}(X)$ we have

$$\dim(\text{Sec}_{\bar{h}}(X)) = n\bar{h} + \bar{h} - 1 = N,$$

where $n = \dim(X)$. Furthermore $\dim(\mathcal{I}) = n(h - \bar{h}) + h - \bar{h} = nh - h - (n\bar{h} + \bar{h})$. Now, substituting $n\bar{h} + \bar{h} = N + 1$ we get $\dim(\mathcal{I}) = h(n + 1) - N - 1 = \dim(VSP_H^X(h))$. Therefore χ is dominant and $VSP_H^X(h)$ is unirational. \square

Remark 4.2. Under the hypothesis of Theorem 4.1 the natural map $\text{Sec}_{\bar{h}}(X) \rightarrow \mathbb{P}^N$ is dominant and birational. Then, by [Me2, Theorem 2.1], the variety X is rational.

In the following we consider the classical case $X = V_d^n$ of the Veronese variety. When the secant variety of the Veronese variety fills the projective space there are few cases in which we have the uniqueness of the decomposition.

Theorem 4.3. [Me2, Theorem 1] *Fix integers $d > n > 1$. Then a general homogeneous polynomial $F \in k[x_0, \dots, x_n]_d$ can be expressed as a sum of d -th powers of linear forms in a unique way if and only if $d = 5$ and $n = 2$.*

It is known that a general homogeneous polynomial $F \in k[x_0, \dots, x_n]_d$ admits a canonical decomposition as a sum of d -th powers of linear forms in the following cases.

- $n = 1, d = 2h - 1$ [Sy],
- $n = d = 3, h = 5$ [Sy],
- $n = 2, d = 5, h = 7$ [Hi].

We give very simple and geometrical proofs of these facts.

Proposition 4.4. *Let $F \in k[x_0, x_1]_{2h-1}$ be a general homogeneous polynomial. There exists a unique decomposition of F as sum of h linear forms.*

Proof. Let X be the rational normal curve of degree $2h - 1$ in \mathbb{P}^{2h-1} . Since $\dim(\text{Sec}_h(X)) = h + (h - 1) = 2h - 1$ there exists a decomposition of F .

Suppose that $\{l_1, \dots, l_h\}$ and $\{L_1, \dots, L_h\}$ are two distinct decompositions of F . Let H_l and H_L be the two $(h - 1)$ -planes generated by the decompositions. The point F_{2h-1} belongs to $H_l \cap H_L$. So the linear space $H = \langle H_l, H_L \rangle$ has dimension

$$\dim(H) \leq (h - 1) + (h - 1) = 2h - 2.$$

If $H_l \cap H_L = \{F\}$, then $\dim(H) = (h - 1) + (h - 1) = 2h - 2$. So H is a hyperplane in \mathbb{P}^{2h-1} and $H \cdot X \geq 2h$. A contradiction because $\deg(X) = 2h - 1$.

If H_l and H_L have s common points, then H_l and H_L intersect in $s + 1$ points q_1, \dots, q_s, F . In this case $H_l \cap H_L \cong \mathbb{P}^s$ and $\dim(H) = 2h - 2 - s$. We choose s points p_1, \dots, p_s on X in general position. So $\Pi = \langle H, p_1, \dots, p_s \rangle$ is a hyperplane such that $\Pi \cdot X \geq 2h - s + s = 2h$, a contradiction. We conclude that the decomposition of F in h linear factors is unique. \square

In order to reconstruct the decomposition we consider the following construction.

Construction 4.5. The partial derivatives of order $h - 2$ of F are $\binom{h-2+1}{1} = h - 1$ homogeneous polynomials of degree $h + 1$. Let $\nu_{h+1}^1 : \mathbb{P}^1 \rightarrow \mathbb{P}^{h+1}$ be the Veronese embedding and let $X = \nu_{h+1}^1(\mathbb{P}^1)$ be the corresponding rational normal curve. Consider the projection

$$\pi : \mathbb{P}^{h+1} \dashrightarrow \mathbb{P}^2$$

from the $(h - 2)$ -plane H_∂ spanned by the partial derivatives. Since the decomposition $\{L_1, \dots, L_h\}$ of F is unique, the projection $\bar{X} = \pi(X)$ has a unique singular point $P = \pi(\langle L_1^{h+1}, \dots, L_h^{h+1} \rangle)$ of multiplicity h . Now to find the decomposition we have to compute the intersection $H \cdot X = \{L_1^{h+1}, \dots, L_h^{h+1}\}$, where $H = \langle H_\partial, P \rangle$.

Example 4.6. We consider the polynomial

$$F = x_0^3 + x_0^2 x_1 - x_0 x_1^2 + x_1^3 \in k[x_0, x_1]_3.$$

that is the point $[1 : 1 : 1 : 1] \in \mathbb{P}^3$. The projection from $[1 : 1 : 1 : 1]$ to the plane $\{X = 0\} \cong \mathbb{P}^2$ is given by

$$\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2, [X : Y : Z : W] \mapsto [Y - X : X + Z : W - X].$$

We compute the projection $C = \pi(X)$ of the twisted cubic curve X , and the singular point of C ,

$$P = \text{Sing}(C) = [4 : 10 : 9].$$

The line $L = \langle [1 : 1 : 1 : 1], P \rangle$ is given by the following equations

$$\begin{cases} 3X - 5Y - 2Z = 0, \\ 5 - 9Y + 4W = 0. \end{cases}$$

The intersection $X \cdot L$ is given by

$$\begin{aligned} L_1^3 &= [0.0515957 : 0.4157801 : 1.1168439 : 1], \\ L_2^3 &= [155.0515957 : 86.5842198 : 16.1168439 : 1]. \end{aligned}$$

These points correspond to the linear forms

$$\begin{aligned} L_1 &= -0.3722812x_0 + x_1, \\ L_2 &= 5.3722813x_0 + x_1. \end{aligned}$$

Indeed we have

$$F = 0.99322 \cdot (-0.3722812x_0 + x_1)^3 + 0.00678 \cdot (5.3722813x_0 + x_1)^3.$$

Now we consider Hilbert's theorem.

Theorem 4.7. *Let $F \in k[x_0, x_1, x_2]_5$ be a general homogeneous polynomial. Then F can be decomposed as sum of seven linear forms*

$$F = L_1^5 + \dots + L_7^5.$$

Furthermore the decomposition is unique.

Proof. By the main result in [AH] we have $\dim(VSP(F, 7)) = 0$. Assume that F admits two different decompositions $\{L_1, \dots, L_7\}$ and $\{l_1, \dots, l_7\}$. Consider the second partial derivatives of F . These are six general homogeneous polynomials of degree three. Let $H_\partial \subseteq \mathbb{P}^9$ be the linear space they generate. Clearly a decomposition of F induces a decomposition of its partial derivatives and we have

$$H_L = \langle L_1^3, \dots, L_7^3 \rangle \supset H_\partial \subset \langle l_1^3, \dots, l_7^3 \rangle = H_l.$$

Since F is general both H_L and H_l intersect the Veronese surface $V_3^2 \subseteq \mathbb{P}^9$ at 7 distinct points. Let

$$\pi : \mathbb{P}^9 \dashrightarrow \mathbb{P}^3$$

be the projection from H_∂ , and $\overline{V} = \pi(V_3^2)$. Then \overline{V} is a surface of degree $\deg(\overline{V}) = 9$ with two points of multiplicity 7 corresponding to $\pi(H_L)$ and $\pi(H_l)$. The line $\langle \pi(H_L), \pi(H_l) \rangle$ intersects \overline{V} with multiplicity at least $14 > \deg(\overline{V})$. Then $\langle \pi(H_L), \pi(H_l) \rangle \subset \overline{V}$ and the 7-plane $H := \langle H_L, H_l \rangle$ intersect V_3^2 along a curve Γ corresponding to $\langle \pi(H_L), \pi(H_l) \rangle$. The construction of Γ yields

$$\deg \Gamma \leq (H_L \cdot V_3^2) = 7.$$

On the other hand $\deg \Gamma = 3j$ therefore we end up with the following possibilities.

- If $\deg \Gamma = 3$ then Γ is a twisted cubic curve contained in H and

$$H_l \cdot \Gamma = H_L \cdot \Gamma = 3.$$

We may assume that $H_l \cap \Gamma = \{l_1^3, l_2^3, l_3^3\}$ and $H_L \cap \Gamma = \{L_1^3, L_2^3, L_3^3\}$. Let Λ be the pencil of hyperplanes containing H , and $\nu_3^2 : \mathbb{P}^2 \rightarrow V_3^2$ the Veronese embedding. The linear system $\nu_3^{2*}(\Lambda|_{V_3^2})$ is a pencil of conics and therefore the base locus of $\Lambda|_{V_3^2}$

consists of at least four points. To conclude observe that the base locus of $\Lambda|_{V_3^2}$ contains $H \cap V_3^2$. This forces

$$\{L_4^3, L_5^3, L_6^3, L_7^3\} = \{l_4^3, l_5^3, l_6^3, l_7^3\},$$

and consequently $H_L = H_l$.

- If $\deg \Gamma = 6$ then

$$H_l \cdot \Gamma = H_L \cdot \Gamma = 6.$$

We may assume that $\Gamma \supset \{L_1^3, \dots, L_6^3\} \cup \{l_1^3, \dots, l_6^3\}$. Let Λ be the pencil of hyperplanes containing H . The linear system $\nu_3^{2*}(\Lambda|_{V_3^2})$ is a pencil of lines and therefore the base locus of $\Lambda|_{V_3^2}$ consists of at least one point. This forces

$$L_7^3 = l_7^3,$$

and therefore $H_L = H_l$.

In both cases we have $H_L = H_l$ and so the contradiction $\{L_1, \dots, L_7\} = \{l_1, \dots, l_7\}$. \square

The following construction is inspired by the proof of Theorem 4.7, and provides a method to reconstruct the decomposition.

Construction 4.8. Let $\{L_1, \dots, L_7\}$ be the unique decomposition of F . Consider now the projection

$$\pi : \mathbb{P}^9 \dashrightarrow \mathbb{P}^3$$

from the linear space H_∂ . The image of the Veronese variety \overline{V} is a surface of degree 9 in \mathbb{P}^3 . Furthermore it has a point P of multiplicity 7, which comes from the contraction of $H_L = \langle L_1^3, \dots, L_7^3 \rangle$. This is the unique point of multiplicity 7 on \overline{V} by the uniqueness of the decomposition. From this discussion we derive the following algorithm to find the decomposition.

- Compute the second partial derivative of F .
- Compute the equation of the 5-plane H_∂ spanned by the derivatives.
- Project the Veronese variety V in \mathbb{P}^3 from H_∂ .
- Compute the point $P \in \overline{V}$ of multiplicity 7.
- Compute the 6-plane $H = \langle H_\partial, P \rangle$ spanned by H_∂ and the point P .
- Compute the intersection $V_3^2 \cdot H = \{L_1^3, \dots, L_7^3\}$.

We can prove the uniqueness theorem for $n = 3$, $d = 3$ and $h = 5$ with a variation on the argument we used in the proof of Theorem 4.7. It is enough to consider the first partial derivatives of a general polynomial $F \in k[x_0, x_1, x_2, x_3]_3$ instead of the second partial derivatives. However *G. Ottaviani* pointed out the following very simple proof of Sylvester's pentahedral theorem.

Theorem 4.9. *Let $F \in k[x_0, x_1, x_2, x_3]_3$ be a general homogeneous polynomial. Then F can be decomposed as sum of five linear forms*

$$F = L_1^3 + \dots + L_5^3.$$

Furthermore the decomposition is unique.

Proof. By the main result in [AH] we have $\dim(VSP(F, 5)) = 0$. Let us consider

$$P_\xi F = \xi_0 \frac{\partial F}{\partial x_0} + \xi_1 \frac{\partial F}{\partial x_1} + \xi_2 \frac{\partial F}{\partial x_2} + \xi_3 \frac{\partial F}{\partial x_3}.$$

If $F = L_1^3 + \dots + L_5^3$ then we have

$$P_\xi F = \sum_{i=1}^5 \xi_i \lambda_i L_i^2.$$

Therefore $P_\xi F$ has rank 2 on the points $\xi \in \mathbb{P}^3$ on which three of the linear forms L_i vanish simultaneously. These points are $\binom{5}{3} = 10$.

Now we consider the subvariety X_2 of \mathbb{P}^9 parametrizing rank 2 quadrics. A quadric Q of rank 2 is the union of two planes. We have $\dim(X_2) = 6$. To find the degree of X_2 we have to intersect it with a 3-plane. So the degree of X_2 is equal to the number of quadrics of rank 2 passing through 6 general points of \mathbb{P}^3 . These quadrics are $\frac{1}{2} \binom{6}{3} = 10$. Therefore $\dim(X_2) = 6$ and $\deg(X_2) = 10$. Now, let us consider the 3-plane

$$\Gamma = \{P_\xi F \mid \xi \in \mathbb{P}^3\} \subseteq \mathbb{P}^9.$$

The intersection $\Gamma \cap X_2 = \{P_\xi F \mid \text{rank}(P_\xi F) = 2\}$ is a set of 10 points. These points have to be the 10 points we have found in the first part of the proof. Then the decomposition of F is unique. \square

The argument used in the proof suggests us an algorithm to reconstruct the decomposition.

Construction 4.10. Consider F and its first partial derivatives.

- Compute the 3-plane Γ spanned by the partial derivatives of F .
- Compute the intersection $\Gamma \cdot X_2$, where X_2 is the variety parametrizing the rank 2 quadrics in \mathbb{P}^3 .
- Consider the 10 points in the intersection. On each plane we are looking for there are 6 of these points, furthermore on each plane there are 4 triples of collinear points.

Then with these 10 points we can construct exactly $\frac{\binom{10}{3}}{\binom{6}{3}+4} = 5$ planes. These planes give the decomposition of F . Note that a priori we have $\binom{10}{6} = 210$ choices, but we are interested in sets of six points $\{P_{j_1}, \dots, P_{j_6}\}$ which lie on the same plane. We know that there are exactly five of these. To find the five combinations we use the following Matlab script which constructs a matrix A whose lines are the ten points and then computes the 6×4 submatrices of rank 3 of A .

```
Script. P1 = input('Point 1:');
:
P10 = input('Point 10:');
q = input('Precision:');
A = [P1;P2;P3;P4;P5;P6;P7;P8;P9;P10];
t = 1;
B = [];
for a=1:5,
for b=a+1:6,
```

```

for c=b+1:7,
for d=c+1:8,
for f=d+1:9,
for g=f+1:10,
M = [A(a,:);A(b,:);A(c,:);A(d,:);A(f,:);A(g,:)];
disp(t);
t = t+1;
v = [];
for a1 = 1:3,
for a2 = a1+1:4,
for a3 = a2+1:5,
for a4 = a3+1:6,
v = [v,det([M(a1,:);M(a2,:);M(a3,:);M(a4,:)])];
end; end; end; end;
if abs(v(1))<q,abs(v(2))<q,abs(v(3))<q,abs(v(4))<q,abs(v(5))<q,
abs(v(6))<q,abs(v(7))<q,abs(v(8))<q,abs(v(9))<q,abs(v(10))<q,
abs(v(11))<q,abs(v(12))<q,abs(v(13))<q,abs(v(14))<q,abs(v(15))<q,
B = [B M];
end; end; end; end; end; end;
[n,m] = size(B);
s = 1;
for r=1:4:m-3,
disp('Matrix'), disp(s),
s = s+1;
B(:,r:r+3),
end;

```

In the following we work out Theorem 4.1 in the case $X = V_5^2$ and $h \geq 7$.

Proposition 4.11. *Let $F \in k[x_0, x_1, x_2]_5$ be a general homogeneous polynomial. For any $h \geq 7$ the variety $VSP(F, h)$ is unirational.*

Proof. By Theorem 4.7 $VSP(F, 7)$ is a single point. If $h \geq 8$ consider the incidence variety

$$\begin{array}{ccc}
 \mathcal{I} = \{(l_1, \dots, l_{h-7}, G) \mid G \in \langle F, l_1^5, \dots, l_{h-7}^5 \rangle\} & \subseteq & (\mathbb{P}^2)^{h-7} \times \mathbb{P}^{20} \\
 \downarrow \phi & & \downarrow \psi \\
 (\mathbb{P}^2)^{h-7} & & \mathbb{P}^{20}
 \end{array}$$

The map ϕ is dominant and its general fiber is a linear subspace of dimension $h - 7$ in \mathbb{P}^{20} . Then \mathcal{I} is a rational variety of dimension $2(h - 7) + h - 7 = 3h - 21$.

Let $(l_1, \dots, l_{h-7}, G) \in \mathcal{I}$ be a general point. By Theorem 4.7 the polynomial G admits a unique decomposition $G = L_1^5 + \dots + L_7^5$. Since $G \in \langle F, l_1^5, \dots, l_{h-7}^5 \rangle$ we have $L_1^5 + \dots + L_7^5 = \alpha F + \sum_{i=1}^{h-7} \lambda_i l_i^5$, and

$$F = \frac{1}{\alpha} L_1^5 + \dots + \frac{1}{\alpha} L_7^5 - \sum_{i=1}^{h-7} \frac{\lambda_i}{\alpha} l_i^5.$$

We get a generically finite rational map

$$\begin{array}{ccc} \chi : & \mathcal{I} & \dashrightarrow VSP(F, h) \\ & (l_1, \dots, l_{h-7}, G) & \longmapsto \{L_1, \dots, L_7, l_1, \dots, l_{h-7}\} \end{array}$$

Since $\dim(VSP(F, h)) = 3h - 21 = \dim(\mathcal{I})$ the map χ is dominant and $VSP(F, h)$ is unirational. \square

Remark 4.12. Consider a general homogeneous polynomial $F \in k[x_0, x_1, x_2, x_3]_3$. By Theorem 4.9 F admits a unique decomposition as sum of five powers of linear forms. The argument used in Proposition 4.11 in this case shows that $VSP(F, h)$ is unirational for any $h \geq 5$.

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